Two Results concerning the Small Viscosity Solution of Linear Scalar Conservation Laws with Discontinuous Coefficients.

Bruno Fornet

Two results concerning the small viscosity solution of linear conservation laws with discontinuous coefficients.

B. Fornet[fornet@cmi.univ-mrs.fr] [*†]

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Abstract

In this paper, we consider the vanishing viscosity approach of the linear hyperbolic Cauchy problem in 1-D

$$\begin{cases} \partial_t u + \partial_x (au) = f, & \{t > 0, x \in \mathbb{R}\}, \\ u_{|t=0} = h, \end{cases}$$

when the coefficient $a(t,x)$ is discontinuous across the line $\{x = 0\}$ and smooth on $\{x \neq 0\}$. Two cases are treated: the expansive (or completely outgoing) case where $sign\,(xa(t,x)) > 0$, for all (t,x) in a neighborhood of $\{x = 0\}$, and the compressive case (or completely ingoing) case where $sign\,(xa(t,x)) < 0$, for all (t,x) in a neighborhood of $\{x = 0\}$. In both cases, we show that the solution of the viscous problem converges and selects a well defined "generalized solution". In the expansive case, our first result answers the open question of selecting a unique solution to the hyperbolic problem, answering a question raised in paper [9]. In the compressive case, we show the formation of a Dirac measure in the small viscosity limit. Moreover, the considered problem does not need to be the linearized of a shockwave on a shock front. For both results, a detailed asymptotic analysis is made via the construction of approximate solutions at any order, including a boundary layer analysis. Moreover, both results state not only existence and uniqueness of the solution but its stability, and are new.

[*]LATP, Université de Provence,39 rue Joliot-Curie, 13453 Marseille cedex 13, France.

[†]LMRS, Université de Rouen,Avenue de l'Université, BP.12, 76801 Saint Etienne du Rouvray, France.

1 Introduction.

Consider the conservative 1-D Cauchy problem:

$$
\begin{cases} \partial_t u + \partial_x(a(t,x)u) = f, & x \in \mathbb{R}, \\ u|_{t=0} = h \quad . \end{cases} \tag{1.1}
$$

If a is discontinuous through $\{x = 0\}$, problem (1.1) has no classical sense and a new notion of solution has to be introduced. Several approaches have already been proposed. Among them, renormalized solutions for this sort of problems have been introduced by Diperna and Lions in [3]. In [1] and [2], Bouchut, James and Mancini defines a notion of solution around the parallel study of the conservative problem (1.1) and the associated nonconservative problem:

$$
\begin{cases} \partial_t u + a(t,x)\partial_x u = g, & x \in \mathbb{R}, \\ u|_{t=0} = l \quad . \end{cases} \tag{1.2}
$$

In [9], Poupaud and Rascle proposes a notion of solution based on generalized characteristics in the sense of Filippov.

In this short paper, we will consider the vanishing viscosity approach in the case where $a(t,x)$ is a piecewise smooth function. Let us describe our assumptions. Let $T > 0$ be fixed once for all. We will assume that the coefficient a belongs to the space of infinitely differentiable functions, bounded with all their derivatives: $C_b^\infty([0,T] \times \mathbb{R}^*)$, with $\mathbb{R}^* = \mathbb{R} - \{0\}$. Furthermore, we assume that f belongs to $C_0^\infty([0,T] \times \mathbb{R})$ and h belongs to $C_0^\infty(\mathbb{R})$. As a first step, let us take $a(x) := a_R \mathbf{1}_{x>0} + a_L \mathbf{1}_{x<0}$, where a_L and a_R denote two constants in $\mathbb{R}^*$. Different cases have to be considered depending on the sign of a_L and a_R. Among those cases, the most interesting ones are when a_L and a_R are of opposite sign. If $a_L > 0$ and $a_R < 0$ [resp $a_L < 0$ and $a_R > 0$], the associated problem will fall into what we call the ”ingoing case” [resp ”outgoing case” or ”expansive case”]. Our two results state existence, uniqueness and stability of the solution obtained by vanishing viscous perturbation of (1.1). The first result deals with the expansive case where uniqueness is the main concern whereas the second result deals with the ingoing case where existence is the main concern. Let ε denote a positive real number. Having in mind to make ε tends towards zero, we consider the following viscous perturbation of (1.1):

$$
\begin{cases} \partial_t u^\varepsilon + \partial_x \left(a(t,x)u^\varepsilon\right) - \varepsilon \partial_x^2 u^\varepsilon = f, & x \in \mathbb{R}, \\ u^\varepsilon|_{t=0} = h \quad . \end{cases} \tag{1.3}
$$

We prove then a convergence result stating that the solution u^ε of (1.3) tends towards $\underline{u}$ deduced from an asymptotic analysis of the problem. Naturally, $\underline{u}$ is then what could be called the small viscosity solution of (1.1). In the ingoing case, $\underline{u}$ is a measure-valued solution which coincides with the generalized solution introduced in the already cited papers. But the interesting point is the asymptotic expansion which gives a very precise description of the solution. In the expansive case, the result seems to be completely new, since the main difficulty was to "select" a solution among all possible weak solutions.

2 Viscous treatment of the expansive case.

For our first result, let us consider equation (1.3) in the case where the coefficient a satisfies, for all $t \in [0, T]$,

$$a(t, 0^+) > 0,$$

$$a(t, 0^-) < 0.$$

We will denote by a_R the restriction of a to $\{x > 0\}$ and by a_L the restriction of a to $\{x < 0\}$.

Remark 2.1. *The value of* $a|_{x=0}$ *is of no concern here. Moreover, by taking* $f = 0$, $a_L = -1$ *and* $a_R = 1$, *we recover the singular expansive case given by Poupaud and Rascle as an example in [9].*

Let us define $\underline{u}$ by $\underline{u} := u_R \mathbf{1}_{x \geq 0} + u_L \mathbf{1}_{x<0}$, where (u_R, u_L) is the unique solution of the following problem:

$$\begin{cases} \partial_t u_R + \partial_x(a_R u_R) = f_R, & \{x > 0\}, \\ \partial_t u_L + \partial_x(a_L u_L) = f_L, & \{x < 0\}, \\ u_R|_{x=0} = u_L|_{x=0} = 0, \quad \forall t \in (0, T], \\ u_R|_{t=0} = h_R, u_L|_{t=0} = h_L \quad , \end{cases} \tag{2.1}$$

where f_R [resp h_R] denotes the restriction of f [resp h] to $\{x > 0\}$, and f_L [resp h_L] denotes the restriction of f [resp h] to $\{x < 0\}$. Note well that this problem has a unique solution in $L^2([0, T] \times \mathbb{R})$, which is given on the side $\{x < 0\}$ by:

$$\begin{cases} \partial_t u_L + \partial_x(a_L u_L) = f_L, & \{x < 0\}, \\ u_L|_{x=0} = 0, \quad \forall t \in (0, T], \\ u_L|_{t=0} = h_L \quad , \end{cases}$$

and on the side $\{x>0\}$ by:

$$\begin{cases} \partial_t u_R + \partial_x(a_R u_R) = f_R, & \{x>0\}, \\ u_R|_{x=0} = 0, \quad \forall t \in (0,T], \\ u_R|_{t=0} = h_R \quad . \end{cases}$$

Remark that, in general, $h_R(0) = h_L(0) \neq 0$, and thus the corner compatibilities are not satisfied. Let us compute $\underline{u}$ in the case where $f = 0$. We will first introduce some notations. Let Ω_R be $(0,T) \times \mathbb{R}^{*+}$. Consider now the vector field defined through: $(t,x) \mapsto \partial_t + a_R(t,x)\partial_x$. We will denote by Γ_R the characteristic curve passing through $t=0, x=0$ and tangent to this vector field. A parametrization of Γ_R is given by: $\Gamma_R = \{(t, x_R(t)), t \in (0,T)\}$, where x_R is the solution of the equation:

$$\begin{cases} \dfrac{dx_R}{dt}(t) = a_R(t, x_R(t)), & t \in (0,T), \\ x_R(0) = 0 \quad . \end{cases}$$

Let us denote by $\tilde{a}_R$ an arbitrary smooth extension of a_R to $\{x<0\}$. We define then φ_R as the solution of:

$$\begin{cases} (\partial_t + \tilde{a}_R(t,x)\partial_x)\varphi_R = 0, & (t,x) \in (0,T) \times \mathbb{R}, \\ \varphi_R|_{t=0} = x \quad . \end{cases}$$

The obtained φ_R is in $C^\infty((0,T)\times\mathbb{R})$. Moreover, we have:

$$\Gamma_R = \{(t,x) \in \Omega_R : \varphi_R(t,x) = 0\}.$$

Ω_L, Γ_L and φ_L are defined in a symmetric way and there holds:

$$\Gamma_L = \{(t,x) \in \Omega_L : \varphi_L(t,x) = 0\}.$$

Note well that, by construction of φ_L and φ_R, we have:

Lemma 2.2. *There is c such that, for all $(t,x) \in \Gamma_R$, there holds:*
$|\partial_x \varphi_R(t,x)| \geq c > 0, \quad |\partial_x \varphi_L(t,x)| \geq c > 0.$

Proof.
Differentiating the equation with respect to x, we obtain that $v := \partial_x \varphi_R$ is the solution of the following transport equation:

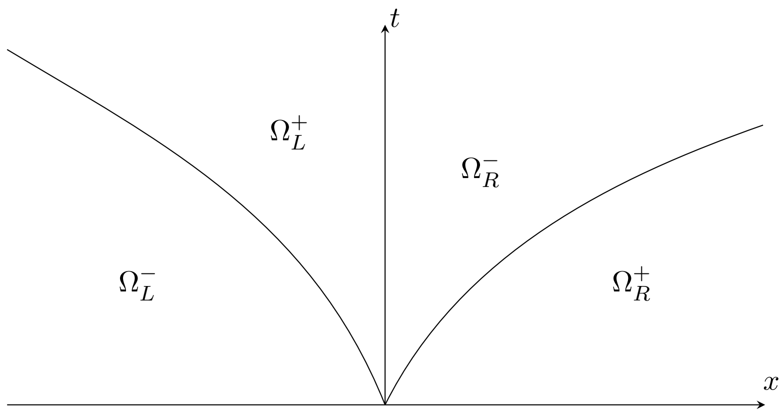
$$\begin{cases} (\partial_t + \tilde{a}_R\partial_x)v + (\partial_x \tilde{a}_R)v = 0, & (t,x) \in (0,T) \times \mathbb{R}, \\ v|_{t=0} = 1 \quad . \end{cases}$$

v is solution of a linear homogeneous equation thus, it cannot cancel without being identically equal to zero along the characteristic curve and in particular for $t = 0$, which achieves to prove our Lemma for φ_R. The proof for φ_L is identical. $\square$

We note for instance:

$$\Omega_L^+ = \{(t, x) \in \Omega_L : \varphi_L(t, x) > 0\},$$

where the "L" stands for "on left hand side of Γ_L" and the $+$ is related to the sign of $\varphi_L(t, x)$. We define in the same manner: Ω_L^-, Ω_R^+ and Ω_R^-.



Let us consider, as an example, the case where the coefficient is piecewise constant and $f = 0$. Solving the limiting hyperbolic problem, we get that, for all $(t, x) \in \Omega_L^+ \bigcup \Omega_R^- \bigcup \{x = 0\}$,

$$\underline{u}(t, x) = 0,$$

for all $(t, x) \in \Omega_R^+$,

$$\underline{u}(t, x) = h_R(x - a_R t),$$

and for all $(t, x) \in \Omega_L^-$,

$$\underline{u}(t, x) = h_L(x - a_L t).$$

Observe that, in this case, the mass of $\underline{u}$ remains constant for all $t \in [0, T]$. Moreover, this example shows clearly the discontinuity of $\underline{u}$ through the lines $\{x - a_R t = 0\}$ and $\{x - a_L t = 0\}$.

Although equation (1.1) trivially admits an infinite number of solutions, we prove the following result:

Theorem 2.3. *There is $C > 0$ such that, for all $0 < \varepsilon < 1$, there holds:*

$$\|u^\varepsilon - \underline{u}\|_{L^\infty([0,T]:L^2(\mathbb{R}))} \leq C\varepsilon^{\frac{1}{4}},$$

where u^ε is the solution of (1.3).

Proof.

We will begin by constructing an approximate solution of problem (1.3). As a first step, we will reformulate problem (1.3) in an equivalent manner. The restrictions of u^ε to $\{x > 0\}$ and $\{x < 0\}$, denoted respectively by u^ε_L and u^ε_R satisfy the following transmission problem:

$$\begin{cases} \partial_t u^\varepsilon_R + \partial_x(a_R u^\varepsilon_R) - \varepsilon\partial_x^2 u^\varepsilon_R = f_R, & \{x > 0\}, t \in [0,T], \\ \partial_t u^\varepsilon_L + \partial_x(a_L u^\varepsilon_L) - \varepsilon\partial_x^2 u^\varepsilon_L = f_L, & \{x < 0\}, t \in [0,T], \\ [u^\varepsilon]_{x=0} = 0, \\ [a(x)u^\varepsilon - \varepsilon\partial_x u^\varepsilon]_{x=0} = 0, \\ u^\varepsilon_R|_{t=0} = h_R, \\ u^\varepsilon_L|_{t=0} = h_L \quad . \end{cases} \tag{2.2}$$

Let us introduce $L^\varepsilon_R = \partial_t + \partial_x(a_R.) - \varepsilon^2\partial_x^2$ and $L^\varepsilon_L = \partial_t + \partial_x(a_L.) - \varepsilon^2\partial_x^2$. We perform the construction of the approximate solution separately on the four domains Ω^-_L, Ω^+_L, Ω^+_R and Ω^-_R. . We will denote by $u^\varepsilon_{app,L,+}$ the restriction of u^ε_{app} to Ω^+_L and so on. Let us present the different profiles and their ansatz:

$$u^\varepsilon_{app,L,+}(t,x) = \sum_{n=0}^{M} \left(\underline{\mathbf{U}}_{L,n,+}(t,x) + \mathbf{U}^c_{L,n,+}\left(t, \frac{\varphi_L(t,x)}{\sqrt{\varepsilon}}\right)\right)\varepsilon^{\frac{n}{2}},$$

where the profiles $\underline{\mathbf{U}}_{n,L,+}$ belongs to $H^\infty(\Omega^+_L)$ and the characteristic boundary layer profiles $\mathbf{U}^c_{n,L,+}(t,x,\theta_L)$ belongs to $e^{-\delta|\theta_L|}H^\infty((0,T) \times \mathbb{R}^{*+})$, for some $\delta > 0$. We will take a similar ansatz for $u^\varepsilon_{app,L,-}$, $u^\varepsilon_{app,R,-}$ and $u^\varepsilon_{app,R,+}$ over their respective domains. Let us explain the different steps of the construction of the approximate solution. We begin by constructing the underlined profiles $\underline{\mathbf{U}}_n$ in cascade, the boundary layer profiles $\mathbf{U}^c_n$ are then computed as a last step. We construct our profiles such that, for all fixed $\varepsilon > 0$, u^ε_{app} belongs to $C^1([0,T] \times \mathbb{R})$. In what follows, we will note:

$$\underline{\mathbf{U}}_{R,j}(t,x) := \underline{\mathbf{U}}_{R,j,+}(t,x)\mathbf{1}_{(t,x)\in\Omega^+_R} + \underline{\mathbf{U}}_{R,j,-}(t,x)\mathbf{1}_{(t,x)\in\Omega^-_R}.$$

Moreover, we will note:

$$\mathbf{U}^c_{R,j}\left(t, x, \frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right) := \mathbf{U}^c_{R,j,+}\left(t, \frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right)\mathbf{1}_{(t,x)\in\Omega^+_R} + \mathbf{U}^c_{R,j,-}\left(t, \frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right)\mathbf{1}_{(t,x)\in\Omega^-_R}.$$

Note well that the dependence of $\mathbf{U}^c_{R,j}$ in x is a bit subtle. Actually, $\mathbf{U}^c_{R,j}$ is piecewise constant with respect to x on each side of Γ_R, which explains that $\mathbf{U}^c_{n,L,+}$ and $\mathbf{U}^c_{n,L,-}$ have no direct dependency in x. Due to their particular meaning, we prefer denoting the profiles $\underline{\mathbf{U}}_{R,0}$ and $\underline{\mathbf{U}}_{L,0}$ by u_R and u_L. Let us note $\mathcal{H}_R$ the differential operator

$$\mathcal{H}_R := \partial_t + \partial_x(a_R.)$$

and $\mathcal{P}_R$ the differential operator

$$\mathcal{P}_R := \partial_t + a_R\partial_x - (\partial_x\varphi)^2\partial^2_{\theta_R} + \partial_x a_R.$$

We have

$$L^\varepsilon_R\ u^\varepsilon_{R,app}\left(t,x,\frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right) = \sum_{j=0}^{M+1} L_{R,j}\left(t,x,\frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right)\varepsilon^{\frac{j}{2}}$$

where

$$L_{R,0} = \mathcal{H}_R u_R + \mathcal{P}_R U^c_{R,0},$$

$$L_{R,1} = \mathcal{H}_R\underline{\mathbf{U}}_{R,1} + \mathcal{P}_R U^c_{R,1} - \left(2(\partial_x\varphi)\partial_x\partial_{\theta_R} + (\partial^2_x\varphi)\partial_{\theta_R}\right)U^c_{R,0},$$

and, for $2 \le j \le M-1$, we get:

$$L_{R,j} = \mathcal{H}_R\underline{\mathbf{U}}_{R,j} + \mathcal{P}_R U^c_{R,j} - \left(2(\partial_x\varphi)\partial_x\partial_{\theta_R} + (\partial^2_x\varphi)\partial_{\theta_R}\right)U^c_{R,j-1} - \partial^2_x\underline{\mathbf{U}}_{R,j-2} - \partial^2_x U^c_{R,j-2},$$

$$L_{R,M} = \mathcal{P}_R U^c_{R,M} - \left(2(\partial_x\varphi)\partial_x\partial_{\theta_R} + (\partial^2_x\varphi)\partial_{\theta_R}\right)U^c_{R,M-1} - \partial^2_x\underline{\mathbf{U}}_{R,M-2} - \partial^2_x U^c_{R,M-2},$$

$$L_{R,M+1} = -\left(2(\partial_x\varphi)\partial_x\partial_{\theta_R} + (\partial^2_x\varphi)\partial_{\theta_R}\right)U^c_{R,M} - \partial^2_x\underline{\mathbf{U}}_{R,M-1} - \partial^2_x U^c_{R,M-1}.$$

Symmetrically, there holds:

$$L^\varepsilon_L\ u^\varepsilon_{L,app}\left(t,x,\frac{\varphi_L(t,x)}{\sqrt{\varepsilon}}\right) = \sum_{j=0}^{M+1} L_{L,j}\left(t,x,\frac{\varphi_L(t,x)}{\sqrt{\varepsilon}}\right)\varepsilon^{\frac{j}{2}}$$

where, for instance, $L_{L,2}$ is given by:

$$L_{L,2} = \mathcal{H}_L\underline{\mathbf{U}}_{L,2} + \mathcal{P}_L U^c_{L,2} - \left(2(\partial_x\varphi_L)\partial_x\partial_{\theta_L} + (\partial^2_x\varphi_L)\partial_{\theta_L}\right)U^c_{L,1} - \partial^2_x u_L - \partial^2_x U^c_{L,0},$$

where $\mathcal{H}_L$ is defined by:

$$\mathcal{H}_L := \partial_t + \partial_x(a_L.)$$

and $\mathcal{P}_L$ is given by:

$$\mathcal{P}_L := \partial_t + a_L\partial_x - (\partial_x\varphi_L)^2\partial^2_{\theta_L} + \partial_x a_L.$$

Plugging $u^\varepsilon_{L,app}$ and $u^\varepsilon_{R,app}$ in the problem (2.2) and identifying the terms with the same scale in ε, making then $|\theta_L|$ and $|\theta_R|$ tend to infinity, we obtain the profiles equations satisfied by the underlined profiles. Let us begin by writing the equations satisfied by $\underline{\mathbf{U}}_{L,j}$ and $\underline{\mathbf{U}}_{R,j}$ for all $0 \leq j \leq M-1$. Thanks to the transmission conditions we had on the viscous problem, we get:

$$\begin{cases} u_{L,+}|_{x=0} - u_{R,-}|_{x=0} = 0, \\ a_L u_{L,+}|_{x=0} - a_R u_{R,-}|_{x=0} = 0. \end{cases}$$

This linear system being invertible, we get then the homogeneous Dirichlet boundary condition:

$$u_L|_{x=0} = u_R|_{x=0} = 0.$$

We can split these equations into three well-posed problems:

$$\begin{cases} \partial_t u_{R,-} + \partial_x(a_R u_{R,-}) = f_{R,-}, & (t,x) \in \Omega^-_R, \\ \partial_t u_{L,+} + \partial_x(a_L u_{L,+}) = f_{L,+}, & (t,x) \in \Omega^+_L, \\ u_{L,+}|_{x=0} = u_{R,-}|_{x=0} = 0, \end{cases}$$

$$\begin{cases} \partial_t u_{R,+} + \partial_x(a_R u_{R,+}) = f_{R,+}, & (t,x) \in \Omega^+_R, \\ u_R|_{t=0} = h_R \quad , \end{cases}$$

$$\begin{cases} \partial_t u_{L,-} + \partial_x(a_L u_{L,-}) = f_{L,-}, & (t,x) \in \Omega^-_L, \\ u_L|_{t=0} = h_L \quad . \end{cases}$$

Since these equations are well-posed, the function $\underline{u}$ is now perfectly defined. Let us go on with the construction of the next profiles. $U_{R,1}$ and $U_{L,1}$ are defined by:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{1,R,-} + \partial_x(a_R \underline{\mathbf{U}}_{1,R,-}) = 0, & (t,x) \in \Omega^-_R, \\ \partial_t \underline{\mathbf{U}}_{1,L,+} + \partial_x(a_L \underline{\mathbf{U}}_{1,L,+}) = 0, & (t,x) \in \Omega^+_L, \\ \underline{\mathbf{U}}_{1,L,+}|_{x=0} = \underline{\mathbf{U}}_{1,R,-}|_{x=0} = 0 \quad . \end{cases}$$

Thus $\underline{\mathbf{U}}_{1,R,-} = 0$ and $\underline{\mathbf{U}}_{1,L,+} = 0$.

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{1,R,+} + \partial_x(a_R \underline{\mathbf{U}}_{1,R,+}) = 0, & (t,x) \in \Omega^+_R, \\ \underline{\mathbf{U}}_{1,R,+}|_{t=0} = 0 \quad , \end{cases}$$

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{1,L,-} + \partial_x(a_L \underline{\mathbf{U}}_{1,L,-}) = 0, & (t,x) \in \Omega^-_{T,L}, \\ \underline{\mathbf{U}}_{1,L,-}|_{t=0} = 0 \quad . \end{cases}$$

Hence $\underline{\mathbf{U}}_{1,R,+} = 0$ and $\underline{\mathbf{U}}_{1,L,-} = 0$. Actually, we see by induction that for all $n \in \mathbb{N}$, we have $\underline{\mathbf{U}}^{\pm}_{2n+1,R,\pm} = 0$ and $\underline{\mathbf{U}}_{2n+1,L,\pm} = 0$. On the other hand for $n \in \mathbb{N}^*$, the profiles $\underline{\mathbf{U}}_{2n,L,\pm}$ and $\underline{\mathbf{U}}_{2n,R,\pm}$ are given by the following well-posed hyperbolic problems.

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{2n,R,-} + \partial_x (a_R \underline{\mathbf{U}}_{2n,R,-}) = \partial_x^2 \underline{\mathbf{U}}_{2n-2,R,-}, & (t,x) \in \Omega^-_{T,R}, \\ \partial_t \underline{\mathbf{U}}_{2n,L,+} + \partial_x (a_L \underline{\mathbf{U}}_{2n,L,+}) = \partial_x^2 \underline{\mathbf{U}}_{2n-2,L,+}, & (t,x) \in \Omega^+_{T,L}, \\ \begin{pmatrix} \underline{\mathbf{U}}_{2n,L,+}|_{x=0} \\ \underline{\mathbf{U}}_{2n,R,-}|_{x=0} \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ -\left(\partial_x \underline{\mathbf{U}}_{2n-2,R,-}|_{x=0} - \partial_x \underline{\mathbf{U}}_{2n,L,+}|_{x=0}\right) \end{pmatrix} \end{cases}$$

where $M := \begin{pmatrix} 1 & -1 \\ a_L|_{x=0} & -a_R|_{x=0} \end{pmatrix}$; remark that the matrix M is nonsingular since $a_L|_{x=0} - a_R|_{x=0} < 0$.

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{2n,R,+} + \partial_x (a_R \underline{\mathbf{U}}_{2n,R,+}) = \partial_x^2 \underline{\mathbf{U}}_{2n-2,R,+}, & (t,x) \in \Omega^+_{T,R}, \\ \underline{\mathbf{U}}_{2n,R,+}|_{t=0} = 0 & . \end{cases}$$

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{2n,L,-} + \partial_x (a_L \underline{\mathbf{U}}_{2n,L,-}) = \partial_x^2 \underline{\mathbf{U}}_{2n-2,L,-}, & (t,x) \in \Omega^-_{T,L}, \\ \underline{\mathbf{U}}_{2n,L,-}|_{t=0} = 0 & . \end{cases}$$

In conclusion, all the profiles $\underline{\mathbf{U}}_n$ are constructed by induction.

We turn now to the construction of the boundary layer profiles $U^c_{L,j,\pm}(t,\theta_L)$ and $U^c_{R,j,\pm}(t,\theta_R)$. We will use the relations imposed on the profiles by the transmission conditions: $[u^\varepsilon_{app}]_{\Gamma_R} = 0$, $[\partial_x u^\varepsilon_{app}]_{\Gamma_R} = 0$, $[u^\varepsilon_{app}]_{\Gamma_L} = 0$, and $[\partial_x u^\varepsilon_{app}]_{\Gamma_L} = 0$; $[u^\varepsilon_{app}]_{\Gamma_R}$ stands for the jump of u^ε_{app} through Γ_R defined, for all $t \in [0,T]$ by:

$$[u^\varepsilon_{app}]_{\Gamma_R}(t) := \lim_{x \to x_R(t), x > x_R(t)} u^\varepsilon_{app}\left(t, x, \frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right) - \lim_{x \to x_R(t), x < x_R(t)} u^\varepsilon_{app}\left(t, x, \frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right),$$

where we recall that $x_R(t)$ is the unique x such that $(t,x) \in \Gamma_R$. $[u^\varepsilon_{app}]_{\Gamma_L}(t)$ is defined the same way. Because u^ε_{app} belongs to $C^1((0,T) \times \mathbb{R}^*)$, for all $0 \le j \le M$, we have:

$$[U^c_{L,j}]_L = -[\underline{\mathbf{U}}_{L,j}]_{\Gamma_L},$$

$$[U^c_{R,j}]_R = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}.$$

Let $[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}$ be given, for all $t \in (0,T)$, by:

$$[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}(t) = \lim_{x \to x_R(t), x > x_R(t)} \underline{\mathbf{U}}_{R,j,+}(t,x) - \lim_{x \to x_R(t), x < x_R(t)} \underline{\mathbf{U}}_{R,j,-}(t,x)$$

and $[U^c_{R,j}]_R$ be defined, for all $t \in (0,T)$, by:

$$[U^c_{R,j}]_R(t) = \lim_{\theta_R \to 0^+} U^c_{R,j,+}(t,\theta_R) - \lim_{\theta_R \to 0^-} U^c_{R,j,-}(t,\theta_R).$$

To avoid writing the exact symmetric equations on $\{x > 0\}$ and $\{x < 0\}$, let us only proceed with the construction of the boundary layer profiles $U^c_{R,j,\pm}$. Referring to the computations above, for all $1 \leq j \leq M+1$, the following quantity must not have any Dirac measure in it:

$$\partial_x \partial_{\theta_R} U^c_{R,j-1} + \frac{1}{2(\partial_x \varphi)} \partial_x(\partial_x(\underline{\mathbf{U}}_{R,j-2} + U^c_{R,j-2})),$$

Our first boundary condition: $[U^c_{L,j}]_L = -[\underline{\mathbf{U}}_{L,j}]_{\Gamma_L}$, ensures that, even if $\partial_x(\underline{\mathbf{U}}_{R,j-2} + U^c_{R,j-2})$ is, in general, discontinuous on Γ_T, it has no Dirac Measure. $\partial_x(\partial_x(\underline{\mathbf{U}}_{R,j-2} + U^c_{R,j-2}))$ is the derivative of such a function and thus has a Dirac Measure. Let us describe this singularity: if we fix $t = t_0$, the Dirac measure forming is

$$\left([\partial_x \underline{\mathbf{U}}_{R,j-2}]|_{x=x_R(t_0)} + [\partial_x U^c_{R,j-2}]_R(t_0)\right) \; \delta_{x=x_R(t_0)}.$$

Hence the Dirac measure forming in $\frac{1}{2(\partial_x \varphi)} \partial_x(\partial_x(\underline{\mathbf{U}}_{R,j-2} + U^c_{R,j-2}))$ is

$$\frac{1}{2(\partial_x \varphi)|_{x=x_R(t_0)}} \left([\partial_x \underline{\mathbf{U}}_{R,j-2}(t_0)]|_{x=x_R(t_0)} + [\partial_x U^c_{R,j-2}(t_0)]_R\right) \; \delta_{x=x_R(t_0)}.$$

where $[\omega]|_{x=x_R(t_0)} = \lim_{x \to x_R(t_0), x > x_R(t_0)} \omega - \lim_{x \to x_R(t_0), x < x_R(t_0)} \omega$.
On the other hand, if $\partial_{\theta_R} U^c_{R,j-1}$ is discontinuous through Γ_R, $\partial_x \left(\partial_{\theta_R} U^c_{R,j-1}\right)$ has a Dirac measure given, for $t = t_0$ by:

$$[\partial_{\theta_R} U^c_{R,j-1}]_R \; \delta_{x=x_R(t_0)}.$$

The game is to construct the boundary layer profiles such that the sum of the two Dirac measures cancel. As a result, the second boundary condition we get is that, $\forall t \in (0,T)$:

$$[\partial_{\theta_R} U^c_{R,j-1}]_R(t) = -\frac{1}{2(\partial_x \varphi)|_{x=x_R(t)}} \left([\partial_x \underline{\mathbf{U}}_{R,j-2}]_{\Gamma_R}(t) + [\partial_x U^c_{R,j-2}(t)]_R\right).$$

The profiles $U^c_{R,0,+}$ and $U^c_{R,0,-}$ are solution of the following heat equation:

$$\begin{cases} \partial_t U^c_{R,0,+} - (\partial_x\varphi_R)^2\partial^2_{\theta_R}U^c_{R,0,+} + (\partial_x a_R)U^c_{R,0,+} = 0 \quad t\in(0,T),\ \{\theta_R>0\}, \\ \partial_t U^c_{R,0,-} - (\partial_x\varphi_R)^2\partial^2_{\theta_R}U^c_{R,0,-} + (\partial_x a_R)U^c_{R,j,-} = 0 \quad t\in(0,T),\ \{\theta_R<0\}, \\ [U^c_{R,0}]_R(t) = -[u_R]_{\Gamma_R}, \quad \forall t\in(0,T), \\ [\partial_{\theta_R}U^c_{R,j}]_R(t) = 0, \quad \forall t\in(0,T), \\ U^c_{R,j,+}|_{t=0} = 0, \\ U^c_{R,j,-}|_{t=0} = 0 \quad . \end{cases}$$

Note well that, since $[u_R]_{\Gamma_R}\neq 0$, the profiles $U^c_{R,0}$ and $U^c_{L,0}$ are not equal to zero.
For all $1\le j\le M$, the profiles $U^c_{R,j,+}$ and $U^c_{R,j,-}$ are given by:

$$\begin{cases} \partial_t U^c_{R,j,+} - (\partial_x\varphi_R)^2\partial^2_{\theta_R}U^c_{R,j,+} + (\partial_x a_R)U^c_{R,j,+} = (\partial^2_x\varphi_R)\partial_{\theta_R}U^c_{R,j-1,+} \quad t\in(0,T),\ \{\theta_R>0\}, \\ \partial_t U^c_{R,j,-} - (\partial_x\varphi_R)^2\partial^2_{\theta_R}U^c_{R,j,-} + (\partial_x a_R)U^c_{R,j,-} = (\partial^2_x\varphi_R)\partial_{\theta_R}U^c_{R,j-1,-} \quad t\in(0,T),\ \{\theta_R<0\}, \\ [U^c_{R,j}]_R(t) = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}, \quad \forall t\in(0,T), \\ [\partial_{\theta_R}U^c_{R,j}]_R(t) = -\dfrac{1}{2(\partial_x\varphi)|_{x=x_R(t)}}\left([\partial_x\underline{\mathbf{U}}_{R,j-1}(t)]_{\Gamma_R}(t) + [\partial_x U^c_{R,j-1}(t)]_R\right), \quad \forall t\in(0,T), \\ U^c_{R,j,+}|_{t=0} = 0, \\ U^c_{R,j,-}|_{t=0} = 0 \quad . \end{cases}$$

Let us now prove the well-posedness of these problems. We take $\psi_{R,j}$ in $H^\infty((0,T)\times\mathbb{R}^*)$ such that

$$[\psi_{R,j}]_R = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R},$$

and

$$[\partial_{\theta_R}\psi_{R,j}]_R(t) = -\frac{1}{2(\partial_x\varphi)|_{x=x_R(t)}}\left([\partial_x\underline{\mathbf{U}}_{R,j-1}(t)]_{\Gamma_R}(t) + [\partial_x U^c_{R,j-1}(t)]_R\right).$$

We can then compute $U^c_{R,j} := U^c_{R,j,+}\mathbf{1}_{\theta_R>0} + U^c_{R,j,-}\mathbf{1}_{\theta_R<0}$ by:

$$U^c_{R,j} := \psi_{R,j} + V^c_{R,j}.$$

$V^c_{R,j}$ is then the solution of the classical heat equation:

$$\begin{cases} \partial_t V^c_{R,j} - (\partial_x\varphi_R)^2\partial^2_{\theta_R})V^c_{R,j} + (\partial_x a_R)V^c_{R,j} = \varphi^*_{R,j}, \quad (t,\theta_R)\in(0,T)\times\mathbb{R}, \\ V^c_{R,j}|_{t=0} = 0 \quad . \end{cases}$$

and $\varphi^*_{R,j}$ is given by:

$$\varphi^*_{R,j} := -\left(\partial_t \psi_{R,j} - (\partial_x \varphi_R)^2 \partial^2_{\theta_R} \psi_{R,j} + (\partial_x a_R)\psi_{R,j}\right) + (\partial^2_x \varphi_R)\partial_{\theta_R} U^c_{R,j-1}.$$

The profiles can thus be constructed by induction using the scheme just introduced.

We will now prove stability estimates.
We define the error $w^\varepsilon := u^\varepsilon_{app} - u^\varepsilon$. Let us denote by $w^{\varepsilon\pm}$ the restriction of w^ε to $\pm x > 0$. $(w^{\varepsilon+}, w^{\varepsilon-})$ is then solution of the transmission problem:

$$\begin{cases} \partial_t w^{\varepsilon+} + \partial_x(a_R w^{\varepsilon+}) - \varepsilon\partial^2_x w^{\varepsilon+} = \varepsilon^M R^{\varepsilon+}, & x > 0, t \in [0,T], \\ \partial_t w^{\varepsilon-} + \partial_x(a_L w^{\varepsilon-}) - \varepsilon\partial^2_x w^{\varepsilon-} = \varepsilon^M R^{\varepsilon-}, & x < 0, t \in [0,T], \\ w^{\varepsilon+}|_{x=0^+} - w^{\varepsilon-}|_{x=0^-} = 0, \\ a_R w^{\varepsilon+}|_{x=0^+} - \varepsilon\partial_x w^{\varepsilon+}|_{x=0^+} = a_L w^{\varepsilon-}|_{x=0^-} - \varepsilon\partial_x w^{\varepsilon-}|_{x=0^-}, \\ w^{\varepsilon+}|_{t=0} = 0, \quad \forall x > 0, \\ w^{\varepsilon-}|_{t=0} = 0, \quad \forall x < 0. \end{cases}$$

By construction of our approximate solution, R^ε belongs to $L^\infty([0,T] : L^2(\mathbb{R}))$. Multiplying by the solution and integrating by parts, we get, for $\{x > 0\}$:

$$\frac{1}{2}\frac{d}{dt}\|w^{\varepsilon+}\|^2_{L^2(\mathbb{R}^*_+)} + \varepsilon\|\partial_x w^{\varepsilon+}\|^2_{L^2(\mathbb{R}^*_+)} - \frac{a_R|_{x=0}}{2}\left(w^{\varepsilon+}|_{x=0}\right)^2 + \varepsilon w^{\varepsilon+}\partial_x w^{\varepsilon+}|_{x=0}$$
$$= \varepsilon^M \int_0^\infty R^{\varepsilon+} w^{\varepsilon+}\, dx - 2\int_0^\infty \partial_x a_R (w^{\varepsilon+})^2\, dx.$$

Note that:

$$-\frac{a_R|_{x=0}}{2}\left(w^{\varepsilon+}|_{x=0}\right)^2 + \varepsilon w^{\varepsilon+}\partial_x w^{\varepsilon+}|_{x=0}$$
$$= \frac{a_R|_{x=0}}{2}\left(w^{\varepsilon+}|_{x=0}\right)^2 - w^{\varepsilon+}|_{x=0}\left(a_R w^{\varepsilon+}|_{x=0} - \varepsilon\partial_x w^{\varepsilon+}|_{x=0}\right).$$

And, for $\{x < 0\}$, we have:

$$\frac{1}{2}\frac{d}{dt}\|w^{\varepsilon-}\|^2_{L^2(\mathbb{R}^*_-)} + \varepsilon\|\partial_x w^{\varepsilon-}\|^2_{L^2(\mathbb{R}^*_-)} + \frac{a_L|_{x=0}}{2}\left(w^{\varepsilon-}|_{x=0}\right)^2 - \varepsilon w^{\varepsilon-}\partial_x w^{\varepsilon-}|_{x=0}$$
$$= \varepsilon^M \int_{-\infty}^0 R^{\varepsilon-} w^{\varepsilon-}\, dx - 2\int_{-\infty}^0 \partial_x a_L (w^{\varepsilon-})^2\, dx.$$

Note that:

$$\frac{a_L|_{x=0}}{2}\left(w^{\varepsilon-}|_{x=0}\right)^2 - \varepsilon w^{\varepsilon-}\partial_x w^{\varepsilon-}|_{x=0}$$
$$= -\frac{a_L|_{x=0}}{2}\left(w^{\varepsilon-}|_{x=0}\right)^2 + w^{\varepsilon-}|_{x=0}\left(a_L w^{\varepsilon-}|_{x=0} - \varepsilon\partial_x w^{\varepsilon-}|_{x=0}\right).$$

Thanks to our boundary condition, there holds:

$$w^{\varepsilon+}|_{x=0}\left(a_R w^{\varepsilon+}|_{x=0} - \varepsilon\partial_x w^{\varepsilon+}|_{x=0}\right) = w^{\varepsilon-}|_{x=0}\left(a_L w^{\varepsilon-}|_{x=0} - \varepsilon\partial_x w^{\varepsilon-}|_{x=0}\right)$$

Thus, by adding our estimates, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|w^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon\|\partial_x w^\varepsilon\|^2_{L^2(\mathbb{R})} + \frac{a_R|_{x=0} - a_L|_{x=0}}{2}\left(w^\varepsilon|_{x=0}\right)^2$$
$$= \varepsilon^M\int_{-\infty}^{\infty} R^\varepsilon w^\varepsilon\ dx - 2\int_0^\infty \partial_x a_R (w^{\varepsilon+})^2\ dx - 2\int_{-\infty}^0 \partial_x a_L (w^{\varepsilon-})^2\ dx.$$

$$\left|\int_{-\infty}^{\infty} R^\varepsilon w^\varepsilon\ dx - 2\int_0^\infty \partial_x a_R (w^{\varepsilon+})^2\ dx - 2\int_{-\infty}^0 \partial_x a_L (w^{\varepsilon-})^2\ dx\right| \le \frac{1}{2}\|R^\varepsilon\|^2_{L^2(\mathbb{R})} + C\|w^\varepsilon\|^2_{L^2(\mathbb{R})},$$

where $C = \frac{1}{2} + 2Max\left(\sup_{(t,x)\in\Omega_L}|\partial_x a_L|, \sup_{(t,x)\in\Omega_R}|\partial_x a_R|\right)$.
Since $a_R|_{x=0} > 0$ and $a_L|_{x=0} < 0$, by Gronwall Lemma, we get the simplified estimate:

$$\|w^\varepsilon\|^2_{L^2(\mathbb{R})}(t) \le \frac{1}{2}\varepsilon^M\int_0^T e^{C(t-s)}\|R^\varepsilon\|^2_{L^2(\mathbb{R})}(s)\ ds.$$

Constructing the profiles up to order $M = 1$, we get then that there is $c > 0$, independent of ε, such that:

$$\|w^\varepsilon\|^2_{L^\infty([0,T]:L^2(\mathbb{R}))} \le c\varepsilon,$$

thus achieving our proof.

□

3 Treatment of the ingoing case.

Let us now introduce our second result. Our second result concerns the case where, for all $t \in [0,T]$, the coefficient a satisfies:

$$a(t, 0^+) < 0,$$

$$a(t, 0^-) > 0.$$

During the study of a similar problem, Poupaud and Rascle show in [9] the formation of a Dirac measure on $\{x = 0\}$ for their solution. We show that a Dirac-measure also forms in the small viscosity limit. We give an asymptotic expansion of the solution u^ε of (1.3), which shows explicitely the convergence to the generalized measure-valued solution $\underline{u}$. The main result is stated in Corollary 3.3 . The problem we consider here appears as one very simple example of the arising of a "δ-measure" in the vanishing viscosity limit. Note that, by using viscous approaches as well, Joseph ([6]) and Tan, Zhang , Zheng ([10]) describe an analogous phenomenon, called δ-shockwave. We will denote by $[\theta]|_{x=0}$ the jump of θ through $\{x = 0\}$ i.e

$$\theta(.,0^+) - \theta(.,0^-).$$

A piecewise smooth u^ε is solution of (1.3) iff its restrictions to $\pm x > 0$ satisfies the equation on $\pm x > 0$ and

$$[a(.,x)u^\varepsilon - \varepsilon\partial_x u^\varepsilon]|_{x=0} = 0,$$

which is the corresponding Rankine-Hugoniot condition. The hyperbolic-parabolic problem (1.3) reformulates then as the following transmission problem:

$$(3.1) \qquad \begin{cases} \partial_t u^{\varepsilon+} + \partial_x(a^+ u^{\varepsilon+}) - \varepsilon\partial_x^2 u^{\varepsilon+} = f^+, & \{x > 0\}, t \in [0;T], \\ \partial_t u^{\varepsilon-} + \partial_x(a^- u^{\varepsilon-}) - \varepsilon\partial_x^2 u^{\varepsilon-} = f^-, & \{x < 0\}, t \in [0;T], \\ u^{\varepsilon+}|_{x=0^+} - u^{\varepsilon-}|_{x=0^-} = 0, & \\ a^+ u^{\varepsilon+}|_{x=0^+} - \varepsilon\partial_x u^{\varepsilon+}|_{x=0^+} = a^- u^{\varepsilon-}|_{x=0^-} - \varepsilon\partial_x u^{\varepsilon-}|_{x=0^-}, & \\ u^{\varepsilon+}|_{t=0} = h^+, & \\ u^{\varepsilon-}|_{t=0} = h^- \quad , & \end{cases}$$

with $u^{\varepsilon+} = u^\varepsilon|_{x>0}$, $a^+ = a|_{x>0}$, $f^+ = f|_{x>0}$, $h^+ = h|_{x>0}$ and $u^{\varepsilon-} = u^\varepsilon|_{x<0}$, $a^- = a|_{x<0}$, $f^- = f|_{x<0}$, $h^- = h|_{x<0}$. Problem (3.1) can be reformulated as the doubled problem on a half-space:

$$(3.2) \qquad \begin{cases} \partial_t \tilde{u}^\varepsilon + \partial_x(\tilde{A}\tilde{u}^\varepsilon) - \varepsilon\partial_x^2 \tilde{u}^\varepsilon = \tilde{f}(t,x), & \{x > 0\}, t \in [0;T], \\ \mathcal{M}_c \tilde{u}^\varepsilon|_{x=0} = 0, & \\ \tilde{u}^\varepsilon|_{t=0} = \tilde{h} \quad . & \end{cases}$$

Let us precise how problem (3.2) is deduced from problem (3.1): $\tilde{u}^\varepsilon$ is a two dimensional vector which first component [resp second component] is $u^{\varepsilon+}(t,x)$ [resp $u^{\varepsilon-}(t,-x)$]. $\tilde{A}$ is defined by:

$$\tilde{A}(t,x) = \begin{bmatrix} a^+(t,x) & 0 \\ 0 & -a^-(t,-x) \end{bmatrix},$$

and $\mathcal{M}_c$ is given as follow:

$$\mathcal{M}_c = \begin{bmatrix} 1 & -1 \\ a^+(t,0) - \varepsilon\partial_x & -a^-(t,0) - \varepsilon\partial_x \end{bmatrix}.$$

In order to prove our main result, there will be two steps: first, we will construct formally an approximate solution of the mixed parabolic problem (3.2) then validate it through the adequate energy estimates. Let us detail the form of our approximate solution, $\tilde{u}^\varepsilon_{app}$ will be constructed as a WKB expansion up to order M of the form:

$$\tilde{u}^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \sum_{n\geq -1} \varepsilon^n \, \mathbf{U}_n\big(t,x,x/\varepsilon\big), \tag{3.3}$$

where $\mathbf{U}_n$ belongs to the space of profiles $\mathcal{P}^*$. Let us define $\mathcal{P}^*$: $\mathbf{U}_n(t,x,z)$ (z is the fast variable x/ε) belongs to $\mathcal{P}^*$ iff it writes:

$$\mathbf{U}_n(t,x,z) = \underline{\mathbf{U}}_n(t,x) + \mathbf{U}^*_n(t,z)$$

with $\underline{\mathbf{U}}_n \in H^\infty([0,T]\times\mathbb{R}^*_+)$ and $\mathbf{U}^*_n(t,z) \in e^{-\delta z}H^\infty([0,T]\times\mathbb{R}^*_+)$ for some $\delta > 0$. In addition, we prescribe $\underline{\mathbf{U}}_{-1}(t,x) = 0$ for obvious reasons. For our treatment, we will see that nonconservative hyperbolic problems are easier to deal with than conservative ones. Moreover, under our assumptions on f and h, a nonconservative hyperbolic problem can be obtained by integrating ours, yielding the desired energy estimates. We begin by introducing the integrated equation:

$$\begin{cases} \partial_t v^\varepsilon + a(t,x)\partial_x v^\varepsilon - \varepsilon\partial_x^2 v^\varepsilon = F, & (t,x)\in[0,T]\times\mathbb{R}, \\ v^\varepsilon|_{t=0} = H, \end{cases} \tag{3.4}$$

where F and H are given by:

$$F = F^+ + F^- := \int_{+\infty}^x f(t,y)dy\mathbf{1}_{x>0} + \int_{-\infty}^x f(t,y)dy\mathbf{1}_{x<0},$$

$$H = H^+ + H^- := \int_{+\infty}^x h(y)dy\mathbf{1}_{x>0} + \int_{-\infty}^x h(y)dy\mathbf{1}_{x<0}.$$

Since f belongs to $C_0^\infty([0,T]\times\mathbb{R})$ and h belongs to $C_0^\infty(\mathbb{R})$, we obtain that $F^\pm$ belongs to $H^\infty([0,T]\times\mathbb{R}^*_\pm)$ and $H^\pm$ belongs to $H^\infty(\mathbb{R}^*_\pm)$. By [5], for all fixed $\varepsilon > 0$, the parabolic problem (3.4) has a unique solution:

$$v^\varepsilon \in C([0,T] : L^2(\mathbb{R})).$$

As a result, the solution u^ε of (1.3) satisfies: $u^\varepsilon = \partial_x v^\varepsilon$.

We will now establish Stability estimates for the hyperbolic-parabolic problem (1.3). These estimates will be proved by derivation of the stability estimates holding true for (3.4). Take C_a given by:

$$C_a := 1 + max(\|\partial_x a^+\|_{L^\infty}, \|\partial_x a^-\|_{L^\infty}).$$

We will now prove the following Proposition:

Proposition 3.1. *For all* $0 < \varepsilon < 1$ *and* $t \in [0,T]$*:*

$$\int_0^T e^{-C_a t}\|u^\varepsilon\|^2_{L^2(\mathbb{R})}\, dt \leq \frac{1}{2\varepsilon}\left(\|H\|^2_{L^2(\mathbb{R})} + \int_0^T e^{-C_a t}\|F\|^2_{L^2(\mathbb{R})} dt\right)$$

Proof. The proof unfolds in two main steps. In a first step, stability estimates are established for (3.4). In a second step, exploiting the fact that the solution of problem (1.3) can be obtained by derivation of the solution of problem (3.4), stability estimates on (1.3) are easily deduced from the stability estimates obtained on (3.4). We will rather work on the reformulation of the nonconservative hyperbolic-parabolic problem (3.4) as the doubled problem on a half space:

$$\begin{cases} \partial_t \tilde{v}^\varepsilon + \tilde{A}\partial_x \tilde{v}^\varepsilon - \varepsilon \partial_x^2 \tilde{v}^\varepsilon = \tilde{F}(t,x), \qquad x > 0,\ t \in [0,T], \\ \mathcal{M}_{nc}\tilde{v}^\varepsilon|_{x=0} = 0, \\ \tilde{v}^\varepsilon|_{t=0} = \tilde{H}, \end{cases} \tag{3.5}$$

with, for all $x > 0$ and $t \in [0,T]$:

$$\tilde{v}^\varepsilon(t,x) = \left(\begin{array}{c} \tilde{v}^{\varepsilon+}(t,x) := v^\varepsilon(t,x) \\ \tilde{v}^{\varepsilon-}(t,x) := v^\varepsilon(t,-x) \end{array} \right), \quad \tilde{F} = \left(\begin{array}{c} F^+(t,x) \\ F^-(t,-x) \end{array} \right), \quad \tilde{H} = \left(\begin{array}{c} H^+(t,x) \\ H^-(t,-x) \end{array} \right),$$

$$\tilde{A} = \left[\begin{array}{cc} a^+(t,x) & 0 \\ 0 & -a^-(t,-x) \end{array} \right] \quad and \quad \mathcal{M}_{nc} = \left[\begin{array}{cc} 1 & -1 \\ \partial_x & \partial_x \end{array} \right].$$

Multiplying (3.5) by $\tilde{v}^\varepsilon$ and integrating with respect to x between 0 and ∞ gives, abbreviating $\|.\|_{L^2(\mathbb{R}^*_+)}$ by $\|.\|_{L^2}$

$$\|\tilde{v}^\varepsilon\|^2_{L^2}(t) \leq \int_0^t e^{C_a(t-s)}\|\tilde{F}(s,.)\|^2_{L^2} ds + e^{C_a t}\|\tilde{H}\|_{L^2}$$

This gives that $v^\varepsilon \in L^\infty([0,T] : L^2(\mathbb{R}))$ for all finite time $T > 0$.

Moreover,

$$\frac{d}{dt}\|\tilde{v}^\varepsilon\|_{L^2}^2 + 2\varepsilon\|\partial_x \tilde{v}^\varepsilon\|_{L^2}^2 \leq \|\tilde{F}\|_{L^2}^2 + C_a\|\tilde{v}^\varepsilon\|_{L^2}^2$$

Hence, for all $t \in [0,T]$ and $0 < \varepsilon < 1$:

$$\int_0^T e^{-C_a t}\|\partial_x v^\varepsilon\|_{L^2(\mathbb{R})}^2\, dt \leq \frac{1}{2\varepsilon}\left(\|H\|_{L^2(\mathbb{R})}^2 + \int_0^T e^{-C_a t}\|F\|_{L^2(\mathbb{R})}^2 dt\right)$$

This concludes the proof of Proposition 3.1.

□

Let us now construct an approximate solution u_a^ε of equation (1.3). We will construct an approximate solution of (1.3) at any order, according to ansatz 3.3.
For all $-1 \leq n \leq M$, we adopt the following notations:

$$[\mathbf{U}_n^*]_{z=0} := \mathbf{U}_n^{*+}|_{z=0} - \mathbf{U}_n^{*-}|_{z=0},$$

$$[a^{-1}(\partial_t \mathbf{U}_n^*)]_{z=0} := (a^+)^{-1}(\partial_t \mathbf{U}_n^{*+})|_{z=0} - (a^-)^{-1}(\partial_t \mathbf{U}_n^{*-})|_{z=0},$$

$$[\underline{\mathbf{U}}_n]_{x=0} := \underline{\mathbf{U}}_n^+|_{x=0} - \underline{\mathbf{U}}_n^-|_{x=0},$$

$$[\partial_t \underline{\mathbf{U}}_n]_{x=0} := (\partial_t \underline{\mathbf{U}}_n^+)|_{x=0} - (\partial_t \underline{\mathbf{U}}_n^-)|_{x=0},$$

$$[\partial_x \underline{\mathbf{U}}_n]_{x=0} := (\partial_x \underline{\mathbf{U}}_n^+)|_{x=0} + (\partial_x \underline{\mathbf{U}}_n^-)|_{x=0},$$

$$[a\underline{\mathbf{U}}_n]_{x=0} := a^+\underline{\mathbf{U}}_n^+|_{x=0} - a^-\underline{\mathbf{U}}_n^-|_{x=0}.$$

We will compute the $M+1$ first $\mathbf{U}_j^*$ profiles and the $M+2$ first $\underline{U}_j$ profiles. The boundary conditions $\mathcal{M}_c \tilde{u}_{app}^\varepsilon|_{x=0} = 0$ are translated on the profiles by:

$$\begin{cases} [a\mathbf{U}_n^* - \partial_z \mathbf{U}_n^*]_{z=0} = -[a\underline{\mathbf{U}}_n - \partial_x \underline{\mathbf{U}}_{n-1}]_{x=0}, \\ \mathbf{U}_n^{*+}|_{z=0} - \mathbf{U}_n^{*-}|_{z=0} = -\left(\underline{\mathbf{U}}_n^+|_{x=0} - \underline{\mathbf{U}}_n^-|_{x=0}\right), \end{cases}$$

where $[a\underline{\mathbf{U}}_n - \partial_x \underline{\mathbf{U}}_{n-1}]_{x=0} := a^+\underline{\mathbf{U}}_n^+|_{x=0} - \partial_x \underline{\mathbf{U}}_{n-1}^+|_{x=0} - \left(a^-\underline{\mathbf{U}}_n^-|_{x=0} + \partial_x \underline{\mathbf{U}}_{n-1}^-|_{x=0}\right)$ and $[a\mathbf{U}_n^* - \partial_z \mathbf{U}_n^*]_{z=0} := a^+\mathbf{U}_n^{*+}|_{z=0} - \partial_z \mathbf{U}_n^{*+}|_{z=0} - \left(a^-\mathbf{U}_n^{*-}|_{z=0} + \partial_z \mathbf{U}_n^{*+}|_{z=0}\right)$.
Plugging (3.3) into the equation (3.2) and identifying the terms with same powers in ε gives the following profiles equations: The profiles $\underline{\mathbf{U}}_j$ satisfy

$$\underline{\mathbf{U}}_{-1} = 0,$$

and $\forall 0 \leq n \leq M+1$

$$\begin{cases} \partial_t \underline{\mathbf{U}}_n + \partial_x(\tilde{A}\underline{\mathbf{U}}_n) = \underline{\boldsymbol{\varphi}}_n, \\ \underline{\mathbf{U}}_n|_{t=0} = 0. \end{cases}$$

Notice that $\underline{\boldsymbol{\varphi}}_0 := \tilde{f}$ being known, $\underline{\mathbf{U}}_0$ is deduced from it. $\underline{\boldsymbol{\varphi}}_1 := \partial_x^2 \underline{\mathbf{U}}_0$ is then known, which gives $\underline{\mathbf{U}}_1$, and so on. All the profiles $\underline{\mathbf{U}}_j$ having already be computed above, the profiles $\mathbf{U}_j^*$ are deduced from them as solution of the following well-posed equations:

$$\begin{cases} \partial_z^2 \mathbf{U}_{-1}^* - \partial_z(\tilde{A}\mathbf{U}_{-1}^*) = 0, \\ [\mathbf{U}_{-1}^*]_{z=0} = 0, \\ [a^{-1}(\partial_t \mathbf{U}_{-1}^*)]_{z=0} = [a\underline{\mathbf{U}}_0]_{x=0}, \end{cases}$$

$$\begin{cases} \partial_z^2 \mathbf{U}_0^* - \partial_z(\tilde{A}\mathbf{U}_0^*) = \partial_t \mathbf{U}_{-1}^*, \\ [\mathbf{U}_0^*]_{z=0} = -[\underline{\mathbf{U}}_0]_{x=0}, \\ [a^{-1}(\partial_t \mathbf{U}_0^*)]_{z=0} = [a\underline{\mathbf{U}}_1]_{x=0} - [\partial_x \underline{\mathbf{U}}_0]_{x=0}, \end{cases}$$

and, for all $1 \leq n \leq M$, we have:

$$\begin{cases} \partial_z^2 \mathbf{U}_n^* - \partial_z(\tilde{A}\mathbf{U}_n^*) = \partial_t \mathbf{U}_{n-1}^*, \\ [\mathbf{U}_n^*]_{z=0} = -[\underline{\mathbf{U}}_n]_{x=0}, \\ [a^{-1}(\partial_t \mathbf{U}_n^*)]_{z=0} = [a\underline{\mathbf{U}}_{n+1}]_{x=0} - [\partial_x \underline{\mathbf{U}}_n]_{x=0}. \end{cases}$$

To sum up, we have constructed $\tilde{u}_{app}^\varepsilon$ as a finite expansion of the form 3.3 satisfying:

$$\begin{cases} \partial_t \tilde{u}_{app}^\varepsilon + \partial_x(\tilde{A}\tilde{u}_{app}^\varepsilon) - \varepsilon \partial_x^2 \tilde{u}_{app}^\varepsilon = \tilde{f}(t,x) + \varepsilon^M R^\varepsilon, \ (t,x) \in [0;T] \times \mathbb{R}_+^*, \\ \mathcal{M}_c \tilde{u}_{app}^\varepsilon|_{x=0} = 0, \\ \tilde{u}_{app}^\varepsilon|_{t=0} = \tilde{h} \quad , \end{cases}$$

where $\varepsilon^M R^\varepsilon$ is the error we have generated, substituting $\tilde{u}^\varepsilon$ by $\tilde{u}_{app}^\varepsilon$.
Let us denote

$$u_a^\varepsilon\left(t, x, \frac{x}{\varepsilon}\right) = \varepsilon^{-1}\mathbf{U}_{-1}^*\left(t, \frac{x}{\varepsilon}\right) + \mathbf{U}_0^*\left(t, \frac{x}{\varepsilon}\right) + \underline{\mathbf{U}}_0(t,x).$$

This is an approximate solution for $M = 1$.

Theorem 3.2. *Assume that $f \in C_0^\infty([0,T] \times \mathbb{R})$ and $h \in C_0^\infty(\mathbb{R})$, then there is a constant $C > 0$, such that, for all $0 < \varepsilon < 1$:*

$$\int_0^T e^{-C_a t} \|u^\varepsilon - u_a^\varepsilon\|_{L^2(\mathbb{R})}^2 \, dt \leq C\varepsilon.$$

Proof. We denote by $w^{\varepsilon\pm}(t,x) = u^{\varepsilon\pm}_{app}(t,\pm x) - u^{\varepsilon\pm}(t,\pm x)$. By linearity, $w^{\varepsilon\pm}$ satisfies the equation:

$$\begin{cases} \partial_t w^{\varepsilon\pm} + a^{\pm}\partial_x w^{\varepsilon\pm} - \varepsilon\partial_x^2 w^{\varepsilon\pm} = \varepsilon^M R^{\varepsilon\pm}, \quad \{\pm x > 0\}, t \in [0;T] \\ w^{\varepsilon+}|_{x=0} - w^{\varepsilon-}|_{x=0} = 0, \\ \left(a^+ w^{\varepsilon+} - \varepsilon\partial_x w^{\varepsilon+}\right)|_{x=0^+} - \left(a^- w^{\varepsilon-} - \varepsilon\partial_x w^{\varepsilon-}\right)|_{x=0^-} = 0, \\ w^{\varepsilon\pm}|_{t=0} = 0 \quad . \end{cases}$$

We denote $I(R^\varepsilon) := \int_{-\infty}^x R^\varepsilon(t,y)dy\mathbf{1}_{x<0} + \int_{\infty}^x R^\varepsilon(t,y)dy\mathbf{1}_{x>0}$. We can perform the construction of an approximate solution whose restriction to $\pm x > 0$ belongs to $H^\infty([0,T]\times\mathbb{R}^*_\pm)$. $I(R^\varepsilon)$ is a linear combination of the profiles involved in this construction thus belonging to $H^\infty([0,T]\times\mathbb{R}^*)$. As a consequence of Proposition 3.1 , there holds:

$$\int_0^T e^{-C_a t}\|w^\varepsilon\|^2_{L^2(\mathbb{R})}\, dt \leq \frac{1}{2}\varepsilon^{2M-1}\int_0^T e^{-C_a t}\|I(R^\varepsilon)\|^2_{L^2(\mathbb{R})}dt,$$

which achieves our proof. □

As a Corollary, we obtain the limit of u^ε. Let us note u_0 the function defined by:

$$u_0(t,x) := \underline{\mathbf{U}_0^+}(t,x)\mathbf{1}_{x>0} + \underline{\mathbf{U}_0^-}(t,-x)\mathbf{1}_{x<0},$$

and u_{-1} the function defined by:

$$u_{-1}(t,z) := \mathbf{U}^{*+}_{-1}(t,z)\mathbf{1}_{z\geq 0} + \mathbf{U}^{*+}_{-1}(t,-z)\mathbf{1}_{z<0}.$$

Note that u_{-1} is continuous across $\{z=0\}$.

Corollary 3.3. *When ε tends to zero, u^ε converges in $\mathcal{D}'((0,T)\times\mathbb{R})$ towards $\underline{u}$ which is a measure of the form*

$$\underline{u}(t,.) = C(t)\ \delta_{x=0} + u_0(t,.),$$

where $u_0(t,.)$ is the regular part of the measure, and $C(t)\ \delta_{x=0}$ is the singular part. The function $C(t)$ is

$$C(t) = \int_{\mathbb{R}} u_{-1}(t,y)\ dy.$$

We observe that $\lim_{\varepsilon\to 0^+}\|u^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} = \infty$, and thus there is no constant $C>0$ such that:

$$\|u^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} \leq C\left(\|f\|_{L^2([0,T]\times\mathbb{R})} + \|h\|_{L^2([0,T]\times\mathbb{R})}\right), \quad \forall\varepsilon>0.$$

As a consequence, our parabolic problem does not satisfy the Uniform Evans Condition (if it was the case, uniform L^2 estimates in ε would hold).

References

[1] F. Bouchut, F. James, *One-dimensional transport equations with discontinuous coefficients*, Nonlin. Anal., **32** (1998), 891-933.

[2] F. Bouchut, F. James, S. Mancini, *Uniqueness and weak stability for multi-dimensional transport equations with one-sided Lipschitz coefficient*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), **IV** (2005), 1-25.

[3] R.J. DiPerna, P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **98** (1989), 511-547.

[4] B. Fornet, *Small viscosity solutions for hyperbolic systems with discontinuous coefficients*, Preprint (2007).

[5] Y. Ikeda *The Cauchy Problem of Linear Parabolic Equations with Discontinuous and Unbounded Coefficients*, Nagoya Math. J. , **41**(1971), 33-42.

[6] K.T. Joseph *A Riemann problem whose viscosity solutions contain δ-measures, Asymptotic Analysis* , **7** (1993), 105-120

[7] P.G Lefloch, A.E. Tzavaras, *representation of weak limits and definition of nonconservartive products*, SIAM J. Math. Anal. **30** (1999), 1309-1342.

[8] G. Métivier, *Small Viscosity and Boundary Layer Methods : Theory, Stability Analysis, and Applications*, Birkhauser (2003).

[9] F. Poupaud, M. Rascle, *Measure solutions to the linear multidimensional transport equation with discontinuous coefficients*, Comm. Diff. Equ. **22** (1997), 337-358.

[10] D. Tan, T. Zhang , Y.Zheng *Delta-Shock Waves as Limits of Vanishing Viscosity for Hyperbolic Systems of Conservation Laws*, Journal of Differential Equations, **112**(1994), pp 1-32.